2020 A
Fall 2020

Lecture 12

- Change of Variables $n=3$
- An additional topic

The change of variables formula for dim 3 takes the form:

Let $\Phi : \tilde{\Omega} \to \Omega$ be 1-1, onto, C^1-map with C^1-inverse.

Then $\forall$ F conti in Ω,

$$\iint_{\Omega} F(x,y,z)\, dV(x,y,z)$$
$$= \iint_{\tilde{\Omega}} F(\Phi(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV(u,v,w).$$

The assumption on F can be weakened to allow Φ not 1-1 at some points, curves, or surfaces.

Here
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det J_\Phi ,$$

$$J_\Phi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$ Jacobian matrix of Φ.

the derivation of this formula is similar to the 2-dim case.

Note that $\left|\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\right|$

is the area of the //-gram spanned by (a_1, a_2), (b_1, b_2).

Now
$$\left|\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\right|$$

is the volume of the parallelotope spanned by (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3).

e.g. 5 $\Phi(\rho, \varphi, \theta) \mapsto (x, y, z) = (\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$

$$J\Phi = \begin{pmatrix} \cos\theta\sin\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\sin\theta\cos\varphi & \rho\cos\theta\sin\varphi \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix}$$

$$\left|\frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)}\right| = |\rho^2\sin\varphi| = \rho^2\sin\varphi,$$

which is the old formula.

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An additional topic

We pointed out in Lecture 9 that there are two versions of 1-dim change of variables formulas.

Version I Let $\varphi : [\alpha, \beta] \to [a,b]$ be C^1. For conti fcn f on $[a,b]$,

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt.$$

Version II. Let $\varphi : [\alpha, \beta] \to [a,b]$ be C^1 and increasing/decreasing. Then for conti f on $[a,b]$,

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,|\varphi'(t)|\,dt.$$

It is Version II we have extended to dim 2, 3. Here we consider a higher dim. extension of Version I. The materials are taken from

P. Lax, Change of Variables in Multiple Integrals, The American Mathematical Monthly, vol 106, 497-501, 2013.

Theorem 2 Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-map such that $\Phi(x,y) = (x,y)$ for $x^2+y^2 \geq R^2$. Then

$$\iint_{\mathbb{R}^2} F(x,y)\, dA(x,y) = \iint_{\mathbb{R}^2} F\circ\Phi(u,v) \frac{\partial(x,y)}{\partial(u,v)}\, dA(u,v),$$

for any conti F which vanishes outside some ball.

We remark that a similar theorem holds for all dim ≥ 3.

Since $F \equiv 0$ in $(x,y) : x^2+y^2 \geq R^2$, the above integration is over some large ball / rectangle.

Proof: Fix a large $a > 0$ so that outside the rectangle $[-a,a]\times[-a,a]$, Φ is the identity map. Also $F \equiv 0$ outside $[-a,a]\times[-a,a]$. Define

$$G(x,y) = \int_{-a}^{y} F(x,t)\,dt \quad \text{so that} \quad \frac{\partial G}{\partial y}(x,y) = F(x,y).$$

Now,

$$\iint_{\mathbb{R}^2} F(\Phi(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, du\,dv$$

$$= \int_{-a}^{a}\int_{-a}^{a} \frac{\partial G}{\partial y}(x(u,v), y(u,v)) \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} du\,dv$$

$$= \int_{-a}^{a}\int_{-a}^{a} \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial G}{\partial y}(..)\frac{\partial y}{\partial u} & \frac{\partial G}{\partial y}(..)\frac{\partial y}{\partial v} \end{bmatrix} du\,dv$$

$$= \int_{-a}^{a}\int_{-a}^{a} \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial G}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial G}{\partial x}\frac{\partial x}{\partial u} & \frac{\partial G}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial G}{\partial x}\frac{\partial x}{\partial v} \end{bmatrix} du\,dv$$

$$= \int_{-a}^{a}\int_{-a}^{a} \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial}{\partial u} G(x(u,v),y(u,v)) & \frac{\partial}{\partial v} G(x(u,v),y(u,v)) \end{bmatrix} du\,dv$$

$$= \int_{-a}^{a}\int_{-a}^{a} \left[\frac{\partial x}{\partial u}\frac{\partial}{\partial v} G(x(u,v),y(u,v)) - \frac{\partial x}{\partial v}\frac{\partial}{\partial u} G(x(u,v),y(u,v))\right] du\,dv$$

$$= \int_{-a}^{a}\int_{-a}^{a} \frac{\partial x}{\partial u}\frac{\partial}{\partial v} G(x(u,v),y(u,v))\,dv\,du$$

$$- \int_{-a}^{a}\int_{-a}^{a} \frac{\partial x}{\partial v}\frac{\partial}{\partial u} G(x(u,v),y(u,v))\,du\,dv$$

$$= -\int_{-a}^{a}\int_{-a}^{a} \frac{\partial^2 x}{\partial v\partial u} G(x(u,v),y(u,v))\,dv\,du + \int_{-a}^{a} \frac{\partial x}{\partial u} G(x(u,v),y(u,v))\Big|_{v=-a}^{v=a} du$$

$$+ \int_{-a}^{a}\int_{-a}^{a} \frac{\partial^2 x}{\partial u\partial v} G(x(u,v),y(u,v))\,du\,dv$$

$$- \int_{-a}^{a} \frac{\partial x}{\partial v} G(x(u,v),y(u,v))\Big|_{u=-a}^{u=a} dv$$

$(u,a),(u,-a),(a,v),(-a,v)$ are points lying on the rectgle, so $\Phi(u,v)=(u,v)$, and $\frac{\partial x}{\partial u}=1$, $\frac{\partial x}{\partial v}=0$ there. We continue to get

$$= \int_{-a}^{a} \left(G(u,a)-G(u,-a)\right) du$$

$$= \int_{-a}^{a} G(u,a)\, du = \int_{-a}^{a}\int_{-a}^{a} F(u,t)\, dt\, du$$

$$= \iint_{\mathbb{R}^2} F(x,y)\, dx\, dy, \text{ done.}$$

In one step $\partial^2 x/\partial u \partial v$ is involved. It can be removed by some approximation argument.

A beautiful application of theorem 2 is the following Brouwer's fixed point theorem.

Theorem 3. Let B be the ball $\{(x,y) : x^2+y^2 \le 1\}$. A continuous map $G : B \to B$ must admit at least one fixed point.

A fixed pt of a map is a point x s.t. $G(x)=x$.

Fact. Let Φ be the map in theorem 2. It must be onto $\mathbb{R}^2$.

Proof : Suppose not, since Φ is the identity outside some ball; if it is not onto, its target must miss out a small ball B, ie, $\Phi(\mathbb{R}^2) \cap B = \phi$. We fix a continuous function F which is positive in B but $\equiv 0$ outside.

But, look at the change of variables formula:

$$\iint_{\mathbb{R}^2} F(x,y)\, dA(x,y) = \iint_{\mathbb{R}^2} F(\Phi(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, dA(u,v)$$

LHS :

$$\iint_{\mathbb{R}^2} F(x,y)\, dA(x,y) = \iint_{B} F(x,y)\, dA(x,y) > 0$$

RHS :

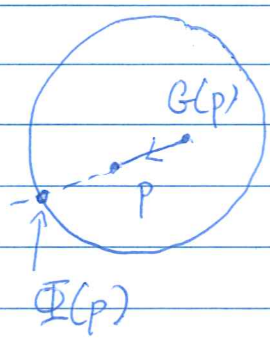
$$\iint_{R^2} F(\Phi(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, dA(u,v) = 0 \quad \because \Phi(\mathbb{R}^2) \cap B = \phi \text{ \& } F = 0 \text{ outside } B$$

Contradiction holds.

Proof of Brouwer's fixed point theorem.

Suppose on the contrary, $\forall\, p = (x,y) \in B$, $G(p) \neq p$,

We extend the line segment $\overrightarrow{G(p)\,p}$ to hit the boundary of B at $\Phi(p)$. Clearly,



$$p \longmapsto \Phi(p)$$

is continuous and $\Phi(p) = p$, $\forall\, p \in$ boundary of B. We extend Φ to $\mathbb{R}^2$ by setting $\Phi(p) = p$, $\forall\, p$ outside B.

when Φ is C^1, we can use the fact above that $\Phi(\mathbb{R}^2) = \mathbb{R}^2$, but by construction $\Phi(B) =$ boundary of B, so $\Phi(\mathbb{R}^2) = \mathbb{R}^2 \setminus$ interior of the ball, contradiction holds.

In general, we may use a sequence Φ_n, C^1-maps, to approximate Φ. Then the fixed points of Φ_n would have a subsequence converging to a fixed point of Φ. You can easily make this clear after taking MATH 2050.

Look up Wiki for more background information for this fundamental theorem in algebraic topology.